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Multistate Reliability Models: A Survey

by

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Department of Statistics  
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Technical Report No. 123  
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# Multistate Reliability Models: A Survey

by

Emad El-Newehi and Frank Proschan

## Abstract

For a long time, the vast majority of the models in reliability theory have concentrated on the case in which both components and systems assume only two possible states; functioning and failed. Unfortunately, this represents a gross oversimplification of the many real life situations in which both components and systems actually assume a variety of states ranging from perfect operation to complete failure.

More recently, papers have appeared which treat the more sophisticated and more realistic situations in which components and systems may assume many states. In the present paper, a survey is made of earlier work, but more especially of some quite recent work (some completed, some still in progress) by the relatively small number of researchers active in this important new area of reliability. It is becoming apparent that this research will generate results not only of value in reliability applications, but also of independent interest in multivariate statistical analysis.

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## 1. Introduction.

The vast majority of reliability analyses assume that components and system are in either of the two states: functioning or failed. In many situations one is capable of distinguishing between various "levels of performance" for both the system and its components. For such cases, the existing dichotomous model is a gross oversimplification of the real situation, whereas models representing multistate systems and components are much more suitable.

Until recently, very little work has been done on this more general problem of multistate systems. Some earlier work treated only very specialized aspects of multistate systems, but no comprehensive treatment of these models was available. Among the earlier papers are [11], [12], [13], [14], [16], and [17]. However there has been recently a growing interest in this important new area of reliability theory. More sophisticated and comprehensive work on multistate models has been performed by Barlow [2], El-Newehi, Proschan, and Sethuraman [8] and Ross [15]. In this expository paper a survey is made of the various treatments of multistate models. We briefly mention the earlier work, but we concentrate on the more recent and more comprehensive treatments of multistate models performed by the relatively small number of researchers active in this important area of reliability.

We now summarize the contents of this paper. Our terminology and notation are similar to that of Barlow and Proschan [1] for the two state case. In Section 2 we present the notation and terminology used throughout the paper. In Section 3 we consider a system of  $n$  components. For each component and for the system itself, we can distinguish among different "levels of performance" represented by a state space  $S$ . For component  $i$ ,  $x_i$  denotes the corresponding state,  $i=1, \dots, n$ , the vector  $\underline{x} = (x_1, \dots, x_n)$  denotes the vector of states of

components  $1, \dots, n$ . The state of the system is assumed to be a deterministic function of  $\phi$  of the states of the components from  $S^n$ , the  $n$ th Cartesian power of  $S$ , into  $S$ . Thus  $\phi(\underline{x})$  is the state of the system corresponding to the component state vector  $\underline{x}$ . We then survey the different choices of state space  $S$ , and various definitions of the structure function  $\phi$  presented in different treatments of multistate models. We investigate structural properties of the various models, occasionally comparing and contrasting them.

In Section 4 we investigate probabilistic aspects of multistate models. We survey the relationship (in a probabilistic sense) between the performance of the system and the performances of its components. For instance, system performance is, as expected, a monotone function of component performances. When the exact value of system performance is difficult to compute, bounds are provided.

In Section 5, we survey dynamic aspects of multistate systems. In earlier sections, it is tacitly assumed that time is fixed. In Section 5, multistate systems are viewed as operating over time. At time 0, the system and each of its components are at the maximal "level of performance". As time passes, the performance levels of components (and consequently of the system) deteriorate to successively lower levels until finally level 0 (complete failure) is reached. Concepts of IFRA and NBU stochastic processes, analogous to the corresponding lifelength distributions in the binary case are defined and studied by various researchers. Some generalized IFRA and NBU closure theorems are presented.

Finally in Section 6, we show by means of two examples how theories of multistate systems may be applied to existing binary reliability models.

## 2. Notation and Terminology.

The vector  $\underline{x} \equiv (x_1, \dots, x_n)$  denotes the vector of states of components 1, ..., n.

$C = \{1, 2, \dots, n\}$  denotes the set of component indices.

$(j_i, \underline{x}) \equiv (x_1, \dots, x_{i-1}, \dots, j, x_{i+1}, \dots, x_n)$  where  $j=0, 1, \dots, M$ .

$(\cdot i, \underline{x}) \equiv (x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n)$

$\underline{j} \equiv (j, \dots, j)$ .

$\underline{y} < \underline{x}$  means that  $y_i \leq x_i$  for  $i=1, \dots, n$  and  $y_i < x_i$  for some  $i$

$\underline{\alpha} = (\alpha_0, \dots, \alpha_M)$  is a probability vector means that  $0 \leq \alpha_j \leq 1$ ,

$$j=0, 1, \dots, M \text{ and } \sum_{j=0}^M \alpha_j = 1.$$

$\underline{\alpha} \leq_{st} \underline{\alpha}'$ , where both  $\underline{\alpha}$ ,  $\underline{\alpha}'$  are probability vectors, means that

$$\sum_{k=j}^M \alpha_k \leq \sum_{k=j}^M \alpha'_k, \quad j=0, 1, \dots, M.$$

$x \vee y$  denotes  $\max(x, y)$ .

$\underline{x} \vee \underline{y} \equiv (x_1 \vee y_1, \dots, x_n \vee y_n)$ .

$x \wedge y$  denotes  $\min(x, y)$ .

$\underline{x} \wedge \underline{y} \equiv (x_1 \wedge y_1, \dots, x_n \wedge y_n)$ .

"Increasing" is used in place of "nondecreasing" and "decreasing"

is used in place of "nonincreasing". When we say  $f(x_1, \dots, x_n)$

is increasing we mean  $f$  is increasing in each argument.

Given a univariate distribution  $F$ , its complement  $1-F$  is denoted by  $\bar{F}$ .

Given a set  $S$ ,  $S^n$  denotes its  $n$ th Cartesian power.



### 3. Deterministic Properties of Multistate Coherent Systems.

Consider a system of  $n$  components. We assume that the performance of the system depends deterministically on the performances of the components. Thus given  $\underline{x}$ , the vector of component states, we may determine  $\phi(\underline{x})$ , the system state. The function  $\phi$  is called the structure function of the system. In the binary case, it is assumed that both components and system are in either of two states: functioning or failed. The variables  $x_i$ ,  $i=1, \dots, n$ , as well as  $\phi(\underline{x})$ , assume their values in the state space  $S = \{0,1\}$ , where 0 denotes failure and 1 denotes functioning. The structure function  $\phi$  is then a map from  $\{0,1\}^n$  into  $\{0,1\}$ . The structure function  $\phi$  satisfies certain conditions that represent intuitively reasonable properties of systems encountered in practice. The following two conditions are required for a binary system to be a coherent structure ([1], Def. 2.1, p. 6),

(i) The function  $\phi(\underline{x})$  is increasing.

(ii) Each component is relevant to the system, i.e., for each  $i$  there exists a vector  $(\cdot i, \underline{x})$  such that  $\phi(1_i, \underline{x}) > \phi(0_i, \underline{x})$ . This means that the function  $\phi$  is not constant in its  $i$ th argument,  $i=1, \dots, n$ .

Condition (i) embodies the reasonable assumption that improving the performance of a component is not harmful to system performance. Condition (ii) eliminates from consideration components which have no effect on system performance. The theory of binary coherent structures has served as a unifying foundation for a mathematical and statistical theory of reliability for the dichotomous case.

The binary model, however, is an oversimplification in describing a situation in which either the system or its components (or both) are capable of assuming a whole range of levels of performance, varying from perfect functioning to complete failure. In these situations, models representing



multistate systems and multistate components are much more useful in describing system performance in terms of component performances. Naturally, the first step in constructing such models is to provide useful definitions of state spaces, representing the sets of levels of performance, and of the structure function  $\phi$ , that relates the performance of the system to the performances of its components. A theory of multistate structures can then serve as a unifying foundation for a mathematical and statistical theory of reliability in the multistate case. Among the earlier attempts to this type, we mention the following two examples:

Hirsch, et al [11], in a treatment of "cannibalization", consider a system of  $n$  components; the state of component  $i$  is represented by the binary variable  $x_i$  assuming the values 1 or 0 according to whether component  $i$  is functioning or failed,  $i=1, \dots, n$ . However the system itself can be in any of  $M+1$  states representing various levels of performance. The set of possible performance levels is assumed to be totally ordered and is then represented, without loss of generality, by  $S = \{0, 1, \dots, M\}$ , where 0 denotes complete failure and  $M$  denotes perfect performance. The structure function  $\phi$  is a map from  $\{0, 1\}^n$  into  $S$ . Note that  $S$  can have at most  $2^n$  elements. The structure function  $\phi$  is assumed to be monotone increasing, with  $\phi(0) = 0$ ,  $\phi(1) = M$ . The authors, however, do not attempt a general treatment of multistate models. Their main concern rather is to investigate the mathematical model for cannibalization to determine how components may be exchanged to improve system performance. The following multistate model is presented by Postelnica [14]: Consider a system of  $n$  components in which the state space for the system and for each of its components is the unit interval  $[0, 1]$ , representing a continuous range of performance from perfect performance (1) to complete failure (0). The structure function  $\phi: [0, 1]^n \rightarrow [0, 1]$  satisfies

the following conditions:

- (a)  $\phi(\underline{1}) = 1$ .
- (b)  $\phi(\underline{0}) = 0$ .
- (c)  $\phi(\underline{x})$  is monotone increasing.
- (d)  $\phi(\underline{c}) \geq c$ ,  $0 \leq c \leq 1$ .
- (e)  $\phi(c, \underline{0}) \leq c$   $0 \leq c \leq 1$ .

The author does not attempt a comprehensive treatment of such multistate structures, but rather investigates some very special applications.

More recent and more comprehensive research in multistate systems has been performed by Barlow [2], El-Newehi, Proschan and Sethuraman [8] (hereafter referred to as EPS [8]), and Ross [15]. The definition given by Barlow [2] for the multistate structure is set-theoretical, based on the concept of min path sets and min cut sets of binary coherent structures. Consider a system of  $n$  components. Assume that the state space for each of the components as well as for the system is the set  $S = \{0, 1, \dots, M\}$ , where 0 denotes the failed state and  $M$  denotes the maximal or perfect state. Let  $P_1, \dots, P_r$  be non-empty subsets of  $C$  such that  $\bigcup_{i=1}^r P_i = C$  and  $P_i \cap P_j = \emptyset$ ,  $i \neq j$ . The structure function  $\phi: S^n \rightarrow S$  is defined by

$$\phi(\underline{x}) = \max_{1 \leq j \leq r} \min_{i \in P_j} x_i, \quad (3.1)$$

where  $\underline{x} \in S^n$  is the vector representing the states of components  $1, 2, \dots, n$ .

In the binary case the structure function given in (3.1) is the most general coherent structure ([1], Chapter 1), and the sets  $P_1, \dots, P_r$  are its min path sets. Let  $\phi'$  be the binary coherent structure associated with  $P_1, \dots, P_r$ . The multistate coherent structure  $\phi$  specified in (3.1) can then be expressed in terms of the corresponding binary coherent structure  $\phi'$  as follows: For each  $i=1, \dots, n$ , let  $y_{ij} = \begin{cases} 1 & \text{if } x_i \geq j \\ 0 & \text{o.w.} \end{cases}$  and let  $\underline{y}_j = (y_{1j}, \dots, y_{nj})$ ,  $j=0, 1, \dots, M$ .

It is fairly easy to see that  $\phi(\underline{x}) \geq j$  iff  $\phi'(y_j) = 1$ , and

$$\phi(\underline{x}) = \sum_{j=1}^M \phi'(y_j). \quad (3.2)$$

Thus the multistate coherent structure given by Barlow [2], is very closely related to a corresponding binary coherent structure. Exploiting this relationship makes it easy to extend results from the binary case to the multistate case.

A more general approach has been taken by EPS [8] to define multistate coherent structures. The common state space for each of the components and for the system is again taken to be the set  $S = \{0, \dots, M\}$ , representing the  $M+1$  levels of performance ranging from complete failure (0) to perfect functioning (M). The structure function  $\phi: S^n \rightarrow S$  is assumed to satisfy three conditions.

Definition 3.1. A system of  $n$  components is said to be a multistate coherent system (MCS) if its structure function  $\phi$  satisfies:

- (i)'  $\phi$  is increasing.
- (ii)' For level  $j$  and component  $i$ , there exists a vector  $(\cdot i, \underline{x})$  such that  $\phi(j_1, \underline{x}) = j$  while  $\phi(l_1, \underline{x}) \neq j$  for  $l \neq j$ ,  $i=1, \dots, n$ , and  $j=0, 1, \dots, M$ .
- (iii)'  $\phi(\underline{j}) = j$  for  $j=0, 1, \dots, M$ .

The three axioms embodied in Definition 3.1 extend the notion of binary coherent system to the new notion of a multistate coherent system. Note that conditions (i)' and (ii)' generalize conditions (i) and (ii) in the binary case. Condition (iii)' is automatically satisfied in the binary case, but is not implied in the present multistate case by (i)' and (ii)'. Also note that since the structure function in (3.1), defined by Barlow [2], satisfy conditions (i)', (ii)', and (iii)' of Definition 3.1, they constitute a subclass of the MCS class. For instance, for a two



component system, only two distinct systems satisfy (3.1), namely, the parallel and the series system, regardless of the cardinality of  $S$ . However for  $S = \{0,1,2\}$ , there are more than 12 MCS's.

The definition given by Ross [16] for a multistate system is less structured than either the Barlow [2] specification or the EPS [8] specification. The state space  $S$  is taken to be  $[0,\infty)$  and the structure function  $\phi$  is any monotone increasing function from  $[0,\infty)^n$  into  $[0,\infty)$ . Ross [15] has not attempted to investigate structural properties of his model; rather, he concentrates on the stochastic properties of his model when observed either at a fixed point in time, or when observed at different points in time (dynamic models). Results of this type will be surveyed in the next two sections.

In the remainder of this section we present various structural properties of the multistate structures given by Barlow [2] and EPS [8]. These properties extend well-known results in the binary case ([1], Chapter 1) to the more general multistate case.

The following theorem gives simple bounds on MCS performance:

Theorem 3.1. Let  $\phi$  be the structure function of an MCS of  $n$  components.

Then

$$\min_{1 \leq i \leq n} x_i \leq \phi(\underline{x}) \leq \max_{1 \leq i \leq n} x_i. \quad (3.3)$$

Theorem 3.1 states that a parallel system yields the best performance of an MCS, and a series system yields the worst performance. Using this theorem, EPS [8] establish probabilistic bounds on system reliability.

As in the binary case, the following lemma in EPS [8] gives a decomposition identity useful in carrying out inductive proofs. It holds for any multistate structure, not just for the MCS.



Lemma 3.1. The following identity holds for any  $n$ -component structure function  $\phi$ :

$$\phi(\underline{x}) = \sum_{j=0}^M \phi(j_1, \underline{x}) I_{[x_1=j]} \text{ for } i=1, \dots, n, \quad (3.4)$$

where

$$I_{[x_1=j]} = \begin{cases} 1 & \text{if } x_1=j \\ 0 & \text{if } x_1 \neq j \end{cases}.$$

As in the binary case, EPS [8], define a dual structure for each multistate structure.

Definition 3.2. Let  $\phi$  be the structure function of a multistate system. The dual structure function  $\phi^D$  is given by:

$$\phi^D(\underline{x}) = M - \phi(M-x_1, M-x_2, \dots, M-x_n). \quad (3.5)$$

It is easy to verify that the dual of an MCS is an MCS.

Example 3.1. The dual of a series (parallel) system is a parallel (series) system. More generally, the dual of a  $k$ -out-of- $n$  is an  $(n-k+1)$ -out-of- $n$  system, where a  $k$ -out-of- $n$  system is given by  $\phi(\underline{x}) = x_{(n-k+1)}$ .

Design engineers have used the well-known principle that redundancy at the component level is preferable to redundancy at the system level. This principle is presented by EPS [8] in mathematical form in (i) of the following theorem; (ii) is a dual result.

Theorem 3.2. Let  $\phi$  be a structure function of an MCS. Then

$$(i) \quad \phi(\underline{x} \vee \underline{y}) \geq \phi(\underline{x}) \vee \phi(\underline{y}).$$

$$(ii) \quad \phi(\underline{x} \wedge \underline{y}) \leq \phi(\underline{x}) \wedge \phi(\underline{y}).$$

Equality holds in (i) for all  $\underline{x}$  and  $\underline{y}$  if and only if the structure is parallel.

Equality holds in (ii) for all  $\underline{x}$  and  $\underline{y}$  if and only if the structure is series.

Parts of (i) and (ii) of Theorem 3.2 are also proved by Barlow [2].

In the binary coherent structures the concepts of minimal path vectors and minimal cut vectors play a crucial role. The analogue in MCS theory is the concept of critical connection vectors. This concept is defined by EPS [8] in the following:

Definition 3.3. A vector  $\underline{x}$  is said to be a connection vector to level  $j$  if  $\phi(\underline{x}) = j$ ,  $j=0,1,\dots,M$ .

Definition 3.4. A vector  $\underline{x}$  is said to be an upper critical connection vector to level  $j$  if  $\phi(\underline{x}) = j$  and  $\underline{y} < \underline{x}$  implies  $\phi(\underline{y}) < j$ ,  $j=1,\dots,M$ .

A lower critical connection vector to level  $j$  can be defined in a dual manner,  $j=0,1,\dots,M-1$ .

The existence of such critical connection vectors is guaranteed by the conditions of Definition 3.1.

Let  $\underline{x}$  be an upper critical connection vector to level  $j$ . Define  $C_j(\underline{x}) = \{i: x_i \geq j\}$ . Obviously  $C_j(\underline{x})$  is a non-empty subset of  $C = \{1,\dots,n\}$ . For  $j=1,\dots,M$ , let  $C_j = \{C_j(\underline{x}): \underline{x} \text{ is an upper critical connection vector to level } j\}$ . Then the following lemma by EPS [8], shows that  $C_j$  enjoys a property similar to that enjoyed by the minimal path sets and minimal cut sets in the binary case.

Lemma 3.2. For  $j=1,\dots,M$ ,

$$UC_j = \{1,\dots,n\}.$$

For  $j=1,\dots,M$ , let  $\underline{y}_1^j, \dots, \underline{y}_{n_j}^j$  be the upper critical connection vectors to level  $j$ , where  $\underline{y}_r^j = (y_{1r}^j, \dots, y_{nr}^j)$ ,  $1 \leq r \leq n_j$ . The following theorem by EPS [8], expresses the state of an MCS using its upper critical connection vectors.

Theorem 3.3. Let  $\phi$  be the structure function of an MCS. Let  $\underline{y}_1^j, \dots, \underline{y}_{n_j}^j$  be its upper critical connection vectors to level  $j$ ,  $j=1,\dots,M$ . Then

$\phi(\underline{x}) \geq j$  if and only if  $\underline{x} \geq \underline{y}_\ell^t$  for some  $j \leq t \leq M$  and some  $1 \leq \ell \leq n_t$ .

The above theorem is utilized to establish bounds on the system performance distribution, as will be shown in the next section.

#### 4. Probabilistic Properties of Multistate Coherent Systems.

The deterministic relationships between the performance of a multistate system and that of its components are exploited by the various researchers in the field to investigate the probabilistic properties of multistate systems. In this section we survey important relationships between the stochastic performance of the system and the stochastic performances of its components. These results provide bounds on system performance which are particularly useful when exact system performance is difficult to evaluate.

Let  $X_i$  denote the random state of component  $i$ ,  $i=1, \dots, n$ . Let  $\underline{X} = (X_1, \dots, X_n)$  be the random vector representing the states of components  $1, \dots, n$ , where the  $X_i$ 's are assumed to be stochastically mutually independent. Then  $\phi(\underline{X})$  is the random variable representing the system state, where  $\phi$  is the structure function of the system. Naturally, the random variables assume their values in the state space  $S$  according to certain probability laws. In the model described by Postelnicu [14],  $X_1, \dots, X_n$  as well as  $\phi(\underline{X})$  are distributed in the unit interval  $[0,1]$ , with cumulative distribution functions  $F_1, \dots, F_n$  and  $F$  respectively. Postelnicu [14] discusses briefly bounds on  $F$  in terms of  $F_1, \dots, F_n$ . In the models described by Barlow [2] and EPS [8], the random variables  $X_1, \dots, X_n$  and  $\phi(\underline{X})$  assume their values in  $S = \{0, \dots, M\}$ , with

$$\begin{aligned} P[X_i=j] &= p_{ij} \\ P[X_i \leq j] &= P_i(j) \\ P[\phi(\underline{X})=j] &= p_j \\ P[\phi(\underline{X}) \leq j] &= P(j), \end{aligned} \tag{4.1}$$

$j=0,1,\dots,M$ , and  $i=1,2,\dots,n$ .  $P_i$  represents the performance distribution of



component  $i$ , while  $P$  represents the performance distribution of the system. Clearly,

$$P_i(j) = \sum_{k=0}^j p_{ik},$$

$$P_i(M) = 1,$$

for  $i=1, \dots, n$ . Similar relationships hold for  $P$ . Let  $h = E\phi(\underline{X})$ ; we may express  $h$  as follows:

$$h \equiv h_{\underline{P}}(P_1, \dots, P_n),$$

since  $h$  is a function of the  $P_1, \dots, P_n$ . Alternatively, we may express  $h$  as follows:

$$h = h_{\underline{p}}(p_1, \dots, p_n),$$

where  $\underline{p}_i \equiv (p_{i0}, p_{i1}, \dots, p_{iM})$  for  $i=1, \dots, n$ . In either case, EPS [8] calls  $h$  the performance function of the system.

Using Lemma 3.1, EPS [8], expresses the performance function of a system of components in terms of performance functions of systems of  $n-1$  components. Such a decomposition identity is useful in carrying out a proof by induction and in deriving properties of  $h$ .

Lemma 4.1. The following identity holds for  $h$ :

$$h(p_1, \dots, p_n) = \sum_{j=0}^M p_{1j} h(j_1, p_1, \dots, p_n), \quad i=1, \dots, n, \quad (4.2)$$

where  $h(j_1, p_1, \dots, p_n) = E\phi(j_1, \underline{X})$ .

The following theorem due to EPS [8] shows that  $h$  is strictly increasing in each  $p_{ij}$ ,  $j>0$ . This property generalizes the corresponding well known property of  $h$  in the binary case.



Theorem 4.1. Let  $h(p_1, \dots, p_n)$  be the performance function of an MCS. Let  $0 < p_{ij} < 1$  for  $i=1, \dots, n$ ,  $j=0, 1, \dots, M$ . Then  $h(p_1, \dots, p_n)$  is strictly increasing in  $p_{ij}$ ,  $i=1, \dots, n$ ,  $j=1, \dots, M$ .

Properties of  $h$  as a function of  $P_1, \dots, P_n$  are also investigated by Barlow [2] and EPS [8]. The following theorem due to EPS [8], shows that  $h(P_1, \dots, P_n)$  is monotone increasing with respect to stochastic ordering. A similar result is proved by Barlow [2] for his subclass of the MCS (see (3.1)) using a different proof. The same property is also proved by Ross [15] for his multistate model.

Theorem 4.2. Let  $P_i, P'_i$  be two possible performance distributions for component  $i$ ,  $i=1, \dots, n$ . Assume  $P_i(j) \geq P'_i(j)$  for  $j=0, 1, \dots, M$ ,  $i=1, \dots, n$ . Let  $P(P')$  be the corresponding system performance distribution. Then

$$\begin{aligned} (i) \quad & P(j) \geq P'(j) \text{ for } j=0, 1, \dots, M, \\ (ii) \quad & h(P_1, \dots, P_n) \leq h(P'_1, \dots, P'_n). \end{aligned} \quad (4.3)$$

A useful decomposition identity is given by EPS [8] for  $P[\phi(x) \geq \ell]$ , namely

Theorem 4.3. Let  $\phi$  be a multistate structure function. Then

$$P[\phi(\underline{X}) \geq \ell] = \sum_{j=0}^M p_{1j} P[\phi(j_1, \underline{X}) \geq \ell], \quad \ell=1, \dots, M. \quad (4.4)$$

Relation (4.4) expresses the survival probability of a structure of  $n$  components in terms of survival probabilities of structures of  $n-1$  components.

Using Theorem 3.1, EPS [8] obtain the following useful bounds on  $P$  and  $h$  in terms of  $P_1, \dots, P_n$ :

Let  $P$  be the performance distribution and  $h$  be the performance function of an MCS. Let  $P_i$  be the  $i$ th component performance distribution,  $i=1, \dots, n$ .

Then for  $j=0,1,\dots,M-1$ :

$$\prod_{i=1}^n P_i(j) \leq P(j) \leq 1 - \prod_{i=1}^n \bar{P}_i(j),$$

$$\sum_{j=1}^M \prod_{i=1}^n \bar{P}_i(j-1) \leq h \leq \sum_{j=1}^M [1 - \prod_{i=1}^n P_i(j-1)],$$

where  $\bar{P}_i(j) = 1 - P_i(j)$ .

The concept of upper connection critical vectors introduced by EPS [8] is exploited to establish further bounds on  $P$  and  $h$ . Let  $y_1^j, \dots, y_{n_j}^j$  be the upper critical connection vectors to level  $j$ ,  $j=1, \dots, M$  (see Definition

3.4). Let  $A_r^j$  denote the event  $[\underline{X} \geq y_r^j]$ ,  $r=1, \dots, n_j$ . By Theorem 3.3,

$$P[\phi(\underline{X}) \geq j] = P\left[\bigcup_{t=j}^M \bigcup_{r=1}^{n_t} A_r^t\right].$$

Now using the well known inclusion-exclusion

principle, the authors establish upper and lower bounds on  $P[\phi(\underline{X}) \geq j] = \bar{P}(j-1)$ .

Note that  $P(A_r^j) = P[\underline{X} \geq y_r^j] = \prod_{i=1}^n P[X_i \geq y_{ir}^j]$  for  $1 \leq r \leq n_j$  and  $j=1, \dots, M$ .

An interesting generalization of the Moore-Shannon Theorem [[1], Theorem 5.4] is obtained by Barlow [2] using the close relationship between his definition of a multistate coherent system and that of the binary coherent system. Recall that corresponding to every multistate structure function  $\phi$  defined by Barlow [2], there is a binary coherent structure  $\phi'$  closely related to  $\phi$  (see (3.1) and (3.2)). Let  $h'$  be the binary reliability function associated with  $\phi'$ , i.e.,  $h' = E\phi'(\underline{Y})$ , where  $\underline{Y} = (Y_1, \dots, Y_n)$  is a random vector whose components are Bernoulli random variables. In view of (3.2), it is easily verified that

$$P[\phi(\underline{X}) \geq j] = E\phi'(\underline{Y}_j) = h'(q_j), \quad (4.5)$$

where  $q_j = (q_{1j}, \dots, q_{nj})$ , and  $q_{ij} = \sum_{k=j}^M P_{ik}$ ,  $i=1, \dots, n$ .

Recall that Moore and Shannon show that binary coherent reliability functions are S-shaped in the sense that if all components function with

probability  $p$ , either  $h(p) \geq p$  or  $h(p) \leq p$  for all  $0 \leq p \leq 1$ , or there exists  $0 < p_0 < 1$  such that  $h(p) \leq p$  for  $0 \leq p \leq p_0$ , while  $h(p) \geq p$  for  $1 \geq p \geq p_0$ . Barlow [2] gives a natural generalization of this result to the multistate case with respect to stochastic ordering.

**Theorem 4.4.** Let  $p_i = \underline{\alpha} = (\alpha_0, \dots, \alpha_M)$  for  $i=1, \dots, n$ . Assume  $h'(p_0) = p_0$  ( $0 < p_0 < 1$ ). Let  $\underline{\alpha}^* = (1-p_0, 0, \dots, 0, p_0)$ . Then

$$(a) \quad \underline{\alpha} \leq_{st} \underline{\alpha}^* \quad \text{implies that} \quad \underline{p} \leq_{st} \underline{\alpha},$$

$$(b) \quad \underline{\alpha} \geq_{st} \underline{\alpha}^* \quad \text{implies that} \quad \underline{p} \geq_{st} \underline{\alpha},$$

where  $\underline{p} = (p_0, \dots, p_M)$ ,  $p_j = P[\phi(\underline{X})=j]$ ,  $j=0, \dots, M$ , and  $\underline{\alpha}' \leq_{st} \underline{\alpha}''$  means that  $\sum_{k=j}^M \alpha'_k \leq \sum_{k=j}^M \alpha''_k$ ,  $j=0, 1, \dots, M$ .

Note that (4.5) is central to the proof of the above theorem.

Finally, in the model proposed by Ross [16],  $X_i$ ,  $i=1, \dots, n$ , and  $\phi(\underline{X})$  are non-negative random variables with distribution functions  $F_i$ ,  $i=1, \dots, n$ , and  $F$  respectively. The function  $r(\bar{F}_1, \dots, \bar{F}_n)$  is defined by

$$r(\bar{F}_1, \dots, \bar{F}_n) = E\phi(\underline{X}),$$

where  $\bar{F}_i = 1 - F_i$ ,  $i=1, \dots, n$ .

Using an extension of Lemma 2.3, p. 84, of Barlow and Proschan [1], Ross [15] proves the following:

**Theorem 4.5.** If  $\phi$  is a binary function then

$$r(\bar{F}_1^\alpha, \dots, \bar{F}_n^\alpha) \geq [r(F_1, \dots, F_n)]^\alpha \quad (4.6)$$

for all  $0 \leq \alpha \leq 1$ .

As a consequence of the above theorem, Ross [15] proves:

**Corollary 4.1.** Let  $X_1, \dots, X_n$  be independent IFRA random variables.

Then



$$(a) \sum_{i=1}^n X_i \text{ is IFRA.}$$

$$(b) P\left\{\prod_{i=1}^n X_i > a\alpha^n\right\} \geq (P\left\{\prod_{i=1}^n X_i > a\right\})^\alpha, \quad 0 \leq \alpha \leq 1.$$

Recall that a distribution function  $F$  with  $F(0) = 0$  is said to be an increasing failure rate average (IFRA) distribution if

$$\bar{F}(\alpha x) \geq [\bar{F}(x)]^\alpha \text{ for all } 0 \leq \alpha \leq 1, x \geq 0.$$

Observe that part (a) of Corollary 4.1 represents the well-known property of the closure of the IFRA distributions under the convolution operation.

Ross [15] also utilizes Theorem 4.5 in proving a generalized IFRA closure theorem which is presented in the next section.

##### 5. Dynamic Models for Multistate Coherent Systems.

In previous sections, we consider deterministic and probabilistic properties of multistate systems at a fixed point in time. In this section we survey some dynamic aspects of multistate structures. We now consider multistate system as operating over time. At time 0 the system and each of its components are at their maximal level of performance. As time passes, the performance levels of components (and consequently of the system) deteriorate to lower levels until finally level 0 (complete failure) is reached.

In the binary case, the length of time during which a component (system) functions is called the lifelength of the component (system); each lifelength is a non-negative random variable. The corresponding lifelength distribution has been classified according to various notions of aging. See, e.g., [1]. Two of the important classes of life distributions are the increasing failure rate average (IFRA) class and the new better than used (NBU) class. Closure of these classes under various basic reliability operations, such as



convolution of distributions and formation of binary coherent systems, is demonstrated in [1]. The counterparts of these concepts in the multistate case have been investigated by Barlow [2], EPS [8], and Ross [15].

Let  $\{X_i(t), t \geq 0\}$  denote the decreasing stochastic process representing the state of component  $i$  at time  $t$ , where  $t$  ranges over the non-negative real numbers for  $i=1, \dots, n$ . The stochastic process  $\{\phi(\underline{X}(t)), t \geq 0\}$  is also decreasing and represents the corresponding system state as time varies, where  $\underline{X}(t) = (X_1(t), \dots, X_n(t))$ . The processes  $\{X_i(t), t \geq 0\}$ ,  $i=1, \dots, n$  are assumed to be mutually independent.

In Barlow's model where the state space is  $\{0, 1, \dots, M\}$ , let us call  $\{j, j+1, \dots, M\}$  the "good" states. Assume that  $[P\{X_i(t) \geq j\}]^{1/t}$  is decreasing in  $t \geq 0$  for fixed  $j$ . It is easily verified that  $[P\{\phi(\underline{X}(t)) \geq j\}]^{1/t}$  is decreasing in  $t \geq 0$  for fixed  $j$ . Thus the above result states that if the length of time spent by each component in the "good" states is an IFRA random variable, then the corresponding length of time spent by the multistate system in the "good" states is also an IFRA random variable. In the binary case this represents the so-called IFRA closure (under formation of binary coherent systems) theorem. Note that from (4.5) the proof of the IFRA closure theorem for Barlow's model is immediate.

The following definition is due to Ross [15].

**Definition 5.1.** The stochastic process  $\{X(t), t \geq 0\}$  is said to be an IFRA process if  $T_a = \inf\{t: X(t) \leq a\}$  is an IFRA random variable for every  $a \geq 0$ .

Having introduced this definition, Ross [15] then proves the following generalized IFRA closure theorem.

**Theorem 5.1.** Let  $\{X_i(t), t \geq 0\}$ ,  $i=1, \dots, n$ , be independent IFRA processes and  $\phi$  a multistate structure function. Then  $\{\phi(\underline{X}(t)), t \geq 0\}$  is

an IFRA process.

The crucial tool in proving the above theorem is Theorem 4.5.

Ross [15] also defines an NBU process and proves a generalized NBU closure theorem. First let us recall the definition of an NBU random variable.

Definition 5.2. A non-negative random variable  $Y$  with distribution function  $F$  is said to be new better than used (NBU) if  $\bar{F}(s+t) \leq \bar{F}(s)\bar{F}(t)$  for all  $s \geq 0$ ,  $t \geq 0$ , where  $\bar{F} = 1-F$ .

Now, Ross [15] gives the following definition of an NBU process.

Definition 5.3. The decreasing stochastic process  $\{X(t) \ t \geq 0\}$  is said to be NBU if with probability 1,

$$P\{T_a > s+t \mid X(u) > a, 0 \leq u \leq s\} \leq P\{T_a > t\}$$

for all  $s, t$ ,  $a \geq 0$ , where  $T_a$  is as in Definition 5.1.

Using his definition, Ross proves:

Theorem 5.2. If the component processes are independent NBU processes, then  $\{\phi(X(t)), t \geq 0\}$  is also NBU.

Another definition of an NBU process is given by EPS [8], and then a simple characterization for this NBU process is derived. Using their characterization, they give a simple proof of a generalized NBU closure theorem. The EPS definition of an NBU process is as follows:

Definition 5.4. The stochastic process  $\{X_i(t), t \geq 0\}$  is an NBU stochastic process if  $T_{i,j} = \inf\{t: X_i(t) \leq j\}$  is an NBU random variable for  $j=0, \dots, M-1$ .

Recall that the state space for the EPS [8] model is the set  $\{0, \dots, M\}$ .

The following lemma gives a simple characterization of an NBU process.

Lemma 5.1. The stochastic process  $\{X_i(t), t \geq 0\}$  is NBU if and only if for all  $s \geq 0$ ,  $t \geq 0$ ,

$$X_1(s+t) \stackrel{st}{\leq} \min(X'_1(s), X'_1(t)),$$

where  $X'_1(s)$  and  $X'_1(t)$  are two independent random variables having the same distributions as  $X_1(s)$ ,  $X_1(t)$  respectively.

Using their Lemma 5.1, EPS [8], prove the following generalized NBU closure theorem.

**Theorem 5.3.** Let  $\phi$  be the structure function of an MCS having  $n$  components and  $\{X_i(t), t \geq 0\}$  be the  $i$ th component performance process,  $i=1, \dots, n$ . Let  $\{X_i(t), t \geq 0\}$ ,  $i=1, \dots, n$ , be mutually independent NBU processes. Then  $\{\phi(X(t)), t \geq 0\}$  is an NBU stochastic process.

**Remark 5.1.** The useful characterization of Lemma 5.1, adapted to the binary case, yields a simpler proof of the NBU closure theorem than the proof given in [1].

## 6. Applications of Multistate Reliability Models.

In this section we illustrate by means of two examples that the theory of multistate reliability models provides useful and new treatments of some existing binary reliability models. This shows that not only do the multistate reliability models provide more realistic analyses of many real life situations, but they also permit us to obtain a better understanding and a more efficient treatment of existing models in the two-state case.

Example 1 appears in El-Newehi, [9].

**Example 1:** EPS [7] study the following model. A series-parallel system consists of  $k+1$  subsystems  $C_0, C_1, \dots, C_k$ , also called cut sets. Cut set  $C_i$  contains  $n_i$  components arranged in parallel,  $i=0, 1, \dots, k$ . No two cut sets have a component in common. Components fail one at a time, and after  $t$  components have failed, each of the remaining components is equally likely to fail,  $t=0, 1, \dots$ . The system fails upon failure of any of the cut sets;



cut set fails when all of its components fail. This model has many applications in the study of reliability, extinction of species, inventory depletion, urn sampling, among others.

In part I of [7], EPS study the probability  $P(n_0; \underline{n})$  that the system fails because of a specified cut set, say  $C_0$ , fails first. Several alternative expressions and recurrence relations for this probability are obtained. Some of these formulae are useful in the computation of desired quantities, while others are used to demonstrate qualitative features like monotonicity, Schur-concavity, etc., and derive asymptotic limits. Similar results are also obtained for the more general model in which an "alarm" rings when a cut set size first reaches  $a$ , where  $a$  is a specified positive integer.

In part II of [7], the authors study the probability distribution, frequency function, and failure rate of the lifelength of series-parallel systems, where system lifelength refers to the number of components that have failed at the time of the system failure.

We now show the relationship between the above model and the multistate models. Assume  $n_i = M$ ,  $i=0, \dots, k$ . Let the state of component  $i$  of an MCS be defined to be the number of functioning components in cut set  $C_i$ . Thus the above model may be viewed as a series MCS. Let  $X_1^i, \dots, X_M^i$  be the random variables representing the lifelengths of components in cut set  $C_i$ ,  $i=0, 1, \dots, k$ . For multistate component  $i$ , the lengths of time spent in state  $M, M-1, \dots, 1$  are given by  $X_{(1)}^i, X_{(2)}^i - X_{(1)}^i, \dots, X_{(M)}^i - X_{(M-1)}^i$ ,  $i=0, \dots, k$ , where  $X_{(1)}^i, X_{(2)}^i, \dots, X_M^{(i)}$  are the  $M$  order statistics of  $X_1^i, \dots, X_M^i$ .

Such an identification relating the multistate model and the binary model permits us to answer a host of questions concerning one of the two models using results obtained for the other. For instance one can find the probability that component 0, say, reaches an "alarm" state  $j$ , say, first. Such information

is helpful in planning maintenance and replacement policies

Example 2. Consider a series system of  $n$  binary components. Assume we have  $M-1$  spares for each of the  $n$  components. A failed component is instantaneously replaced by one of its spares. When the original component  $i$  is functioning (and thus none of the spares has been used), we consider that component  $i$  is in state  $M$ . Upon failure of the original component one of its spares is used to replace it, and so the component now enters state  $M-1$ , etc. Thus we can view the system with its spares as a multistate series system. Let  $X_1^i, \dots, X_M^i$  be random variables representing the lifetimes of component  $i$  and of its spares,  $i=1, \dots, n$ . Assume that all the random variables are mutually independent. Obviously, the length of time spent by component  $i$  in a single "state" or in a group of "states" can be expressed in terms of a sum of an appropriate subset of  $X_1^i, \dots, X_M^i$ ,  $i=1, \dots, n$ . Thus we may view a binary system with spares as a multistate system. Again, such an identification is mutually beneficial in the study of both models.

## REFERENCES

- [1] Barlow, R.E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing. Holt, Rinehart and Winston, New York.
- [2] Barlow, R.E. (1978). Coherent systems with multi-state components. Mathematics of Operation Research (to appear).
- [3] Birnbaum, Z.W., Esary, J.D., and Saunders, S.C. (1961). Multicomponent systems and structures and their reliabilities. Technometrics, 3, 55-77.
- [4] Birnbaum, Z.W. and Esary, J.D. (1965). Some inequalities for reliability functions. Proc. of the Fifth Berkeley Symposium on Math. Statistics and Probability, 271-283.
- [5] Birnbaum, Z.W. (1960). On the probabilistic theory of complex structures. Proc. of the Fourth Berkeley Symposium on Math. Statistics and Probability, 49-55.
- [6] Birnbaum, Z.W., Esary, J.D., and Marshall, A.W. (1966). Stochastic characterization of wearout for components and systems. Ann. Math. Statist. 37, 816-825.
- [7] El-Newehi, E., Proschan, F., and Sethuraman, J. (1978). A simple model with applications in structural reliability, extinction of species, inventory depletion, and urn sampling. Advances in Applied Prob. (to appear).
- [8] El-Newehi, E., Proschan, F., and Sethuraman, J. (1978). Multistate coherent systems. J. Appl. Prob. (to appear).
- [9] El-Newehi, E. (1978). On a simple model with applications in structural reliability, extinction of species, inventory depletion, and urn sampling (in preparation).
- [10] Esary, J.D. and Proschan, F. (1963). Coherent structures of non-identical components. Technometrics, 5, 191-209.
- [11] Hirsch, W.M. et al (1968). Cannibalization in multicomponent systems and the theory of reliability. Naval Research Logistics Quarterly, 15, 331-359.
- [12] Hochberg, M. (1973). Generalized multicomponent systems under cannibalization. Naval Research Logistics Quarterly, 20, 585-605.
- [13] Murchland, J.D. (1975). Fundamental concepts and relations for reliability analysis of multistate systems. Reliability and Fault Tree Analysis, ed. by R.E. Barlow, J. Fussell, and N.D. Singpurwalla, 581-618.
- [14] Postelnicu, V. (1970). Nondichotomic multicomponent structures. Bulletin de la Societe des Sciences Mathematiques de la R.S. Roumaine, Tom 14 (62), nv. 2 209-217.



- [15] Ross, S. (1977). Multi-values state component reliability systems. Technical Report, Department of Industrial Engineering and Operations Research, University of California, Berkeley.
- [16] Simon, R.M. (1970). Optimal cannibalization policies for multi-component systems. SIAM J. Appl. Math., 19 700-711.
- [17] Simon, R.M. (1972). The reliability of multicomponent systems subject to cannibalization. Naval Research Logistics Quarterly, 19 1-14.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  For a long time, the vast majority of the models in reliability theory have concentrated on the case in which both components and systems assume only two possible states; functioning and failed. Unfortunately, this represents a gross oversimplification of the many real life situations in which both components and systems actually assume a variety of states ranging from perfect operation to complete failure.  More recently, papers have appeared which treat the more sophisticated and more realistic situations in which components and systems may assume many			



states. In the present paper, a survey is made of earlier work, but more especially of some quite recent work (some completed, some still in progress) by the relatively small number of researchers active in this important new area of reliability. It is becoming apparent that this research will generate results not only of value in reliability applications, but also of independent interest in multivariate statistical analysis.